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Inequalities for the Perron root related to Levinger's theorem

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Abstract

For the Perron roots of square nonnegative matrices A, B, and $A + D^{-1}B^{\mathrm{T}}D$, where D is a diagonal matrix with positive diagonal entries, the inequality

$$\rho(A + D^{-1}B^{\mathrm{T}}D) \geqslant \rho(A) + \rho(B)$$

is proved under the assumption that A and B have a common unordered pair of nonorthogonal right and left Perron vectors. The case of equality is analyzed. The above inequality generalizes the inequality $\rho(\alpha A + (1-\alpha)B^{\mathrm{T}}) \geqslant \alpha\rho(A) + (1-\alpha)\rho(B)$, proved under stronger assumptions by Bapat, and implies a generalization of Levinger's theorem on the monotonicity of the Perron root of a weighted arithmetic mean of a nonnegative matrix and its transpose. Also, for the Perron root

$$\rho\left(A^{(\alpha)} \circ (D^{-1}A^{\mathrm{T}}D)^{(c-\alpha)}\right), \quad c \geqslant 1, \ 0 \leqslant \alpha \leqslant c,$$

of a weighted (entrywise) geometric mean of A and $D^{-1}A^{\mathrm{T}}D$, where $A^{(\alpha)} = \left(a_{ij}^{\alpha}\right)$ and "$\circ$" denotes the Hadamard product, the monotonicity property dual to that asserted by generalized Levinger's theorem is established. © 1998 Elsevier Science Inc. All rights reserved.

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1. Introduction

In [1], R.B. Bapat proved the following result.

Theorem 1 (Bapat [1]). *Let A and B be $n \times n$ nonnegative irreducible matrices that have a common right Perron vector u and a common left Perron vector v. Then, for any α, $0 \leqslant \alpha \leqslant 1$,*

$$\rho\left(\alpha A + (1-\alpha)B^{\mathrm{T}}\right) \geqslant \alpha\rho(A) + (1-\alpha)\rho(B) \tag{1.1}$$

and equality in (1.1) *occurs if and only if the vectors u and v are collinear.*

Recall that, for a square nonnegative matrix A, its right (left) Perron vector is an eigenvector that corresponds to the spectral radius $\rho(A)$, which is also referred to as the Perron root of A.

It was also noted in [1] that inequality (1.1) remains valid if one drops the assumption that A and B are irreducible but require instead that (a) $u^{\mathrm{T}}v$ be positive (this is a necessary requirement) and (b) that the matrix $\alpha A + (1-\alpha)B^{\mathrm{T}}$ have a positive (right or left) Perron vector. As will be shown in Section 2, the second requirement can actually be dropped.

Bapat's theorem implies the following result as a special case.

Theorem 2 ([1]). *Let A be an $n \times n$ nonnegative matrix and let $0 \leqslant \alpha \leqslant 1$. Then*

$$\rho(A) \leqslant \rho\left(\alpha A + (1-\alpha)A^{\mathrm{T}}\right), \tag{1.2}$$

and, furthermore, if A is irreducible and $0 < \alpha < 1$, then equality holds in (1.2) *if and only if any right Perron vector of A is also a left Perron vector of A.*

As is not difficult to ascertain (see [1]), Theorem 2 is actually equivalent to Levinger's theorem recalled below, which was stated without proof in [7]. Another elementary proof of Levinger's theorem based on perturbation theory can be found in [4].

Theorem 3 (Levinger [7]). *Let A be an $n \times n$ nonnegative matrix. Then the function*

$$\rho\left(\alpha A + (1-\alpha)A^{\mathrm{T}}\right)$$

as a function of α is nondecreasing on $[0, 1/2]$ and nonincreasing on $[1/2, 1]$.

In Section 2 of this paper, we extend Theorem 1 by allowing the matrices A and B to have a common unordered pair of nonorthogonal right and left Perron vectors and by replacing B^{T} with a matrix diagonally similar to it. This leads us to an extension of Theorem 2 (See Theorem 6) and a generalization of Levinger's theorem (see Theorem 7).

Section 3 of this paper deals with weighted geometric means of a nonnegative matrix A and a matrix diagonally similar to its transpose, i.e., with matrices of the form

$$A^{(\alpha)} \circ (D^{-1}A^{\mathrm{T}}D)^{(c-\alpha)}, \quad c \geqslant 1,$$

where $A \circ B = (a_{ij}b_{ij})$ denotes the (entrywise) Hadamard product of A and B, and $A^{(\alpha)} = \left(a_{ij}^{\alpha}\right)$. It is shown that the function $\rho\left(A^{(\alpha)} \circ (D^{-1}A^{\mathrm{T}}D)^{(c-\alpha)}\right)$, $c \geqslant 1$, as a function of α is nonincreasing on $[0, c/2]$ and nondecreasing on $[c/2, c]$. This result, dual to Theorem 7, generalizing Levinger's theorem, is derived (in analogy with the additive case) from a particular case of the following theorem established in [3] (see also [5]), which is in a sense dual to Theorem 1.

Theorem 4 ([3]). *Let A and B be $n \times n$ nonnegative matrices. Then, for $c \geqslant 1$,*

$$\rho(A^{(\alpha)} \circ B^{(c-\alpha)}) \leqslant \rho(A)^{\alpha}\rho(B)^{(c-\alpha)}, \quad 0 \leqslant \alpha \leqslant c. \tag{1.3}$$

Furthermore, if $c = 1$ and A, B are both nonzero, then equality holds in (1.3) if and only if there exists a subset $I \subseteq \{1, \ldots, n\}$, $I \neq \emptyset$, such that

(i) *$A[I]$ and $B[I]$ are both irreducible,*

(ii) *$\rho(A[I]) = \rho(A)$ and $\rho(B[I]) = \rho(B)$,*

(iii) *$\exists \gamma < 0$ such that $B[I] = \gamma\Delta^{-1}A[I]\Delta$, where Δ is a diagonal matrix with positive diagonal entries.*

2. Generalizations of Levinger's theorem

The main result of this section is the following generalization of Bapat's theorem.

Theorem 5. *Let A and B be $n \times n$ nonnegative matrices and let*

$$Au = \rho(A)u, \quad A^{\mathrm{T}}v = \rho(A)v. \tag{2.1}$$

Assume that

$$v^{\mathrm{T}}u > 0 \tag{2.2}$$

and that either

$$Bu = \rho(B)u, \quad B^{\mathrm{T}}v = \rho(B)v \tag{2.3}$$

or

$$B^{\mathrm{T}}u = \rho(B)u, \quad Bv = \rho(B)v. \tag{2.4}$$

Then, for the matrix

$$F = A + D^{-1}B^{\mathrm{T}}D, \tag{2.5}$$

where D is a diagonal matrix with positive diagonal entries, the inequality

$$\rho(F) \geqslant \rho(A) + \rho(B) \tag{2.6}$$

is valid. Furthermore, if at least one of the matrices A, B is irreducible, then equality holds in (2.6) *if and only if the vectors Du and v are collinear under the assumption* (2.3) *or $D = \alpha I_n$, where α is a positive constant, under the assumption* (2.4).

The proof of Theorem 5 will be based on three lemmas below. The first one is borrowed from [4], except for the equality case, which can be easily derived from the proof in [4].

Lemma 1 ([4]). *Let A be an $n \times n$ nonnegative matrix and let u and v be positive vectors of dimension n. If*

$$(Au) \circ v = (A^{\mathrm{T}}v) \circ u, \tag{2.7}$$

then, for any positive vectors x and y of the same dimension such that

$$x \circ y = u \circ v, \tag{2.8}$$

the inequality

$$y^{\mathrm{T}}Ax \geqslant v^{\mathrm{T}}Au \tag{2.9}$$

is valid. Further, for a nondiagonal matrix A, equality in (2.9) *occurs if and only if*

$$x = \gamma u \quad \text{and} \quad y = \frac{1}{\gamma}v, \quad \gamma > 0. \tag{2.10}$$

Lemma 2. *Let A be an $n \times n$ nonnegative matrix, and let u and v be nonnegative vectors such that*

$$Au = \rho(A)u, \quad A^{\mathrm{T}}v = \rho(A)v. \tag{2.11}$$

If

$$v^{\mathrm{T}}u > 0, \tag{2.12}$$

then, for the index subset $I = \{i : u_i v_i > 0,\ 1 \leqslant i \leqslant n\}$,

$$A[I]u[I] = \rho(A)u[I], \quad A[I]^{\mathrm{T}}v[I] = \rho(A)v[I],$$

where $A[I]$ denotes the principal submatrix of A made up from the entries of A on the intersections of its rows and columns with indices from I, and $u[I]$, $v[I]$ are the corresponding subvectors of the vectors u and v.

Proof. If both a and v are positive, then the assertion trivially holds for $I = \{1, \ldots, n\}$. Consider the case where u and v contain zero components. Without loss of generality, we may assume that the vectors u and v are of the form

$$u = \begin{bmatrix} o \\ u_2 \end{bmatrix}, \quad v = \begin{bmatrix} o \\ v_2 \end{bmatrix}, \quad \dim u_2 = \dim v_2 \leqslant n,$$

where u_2 and v_2 have no zero components with the same index, i.e., the vector $u_2 + v_2$ is positive. Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

be the corresponding block partitioning of A. Relations (2.11) imply that

$$A_{22}u_2 = \rho(A)u_2, \quad A_{22}^{\mathrm{T}}v_2 = \rho(A)v_2, \tag{2.13}$$

and thus

$$\rho(A) = \rho(A_{22}).$$

Since the vectors u_2 and v_2 have no zero components with the same index, we may further assume (without loss of generality) that

$$u_2 = \begin{bmatrix} 0 \\ \tilde{u}_2 \\ \tilde{u}_3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ 0 \end{bmatrix},$$

where $\tilde{v}_1$ and $\tilde{u}_3$ are either positive or trivial (i.e., of zero dimension), whereas $\tilde{u}_2$ and $\tilde{v}_2$ are both positive. Note that, in view of (2.12), the vectors $\tilde{u}_2$ and $\tilde{v}_2$ are nontrivial. Let

$$A_{22} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} \\ \tilde{A}_{31} & \tilde{A}_{32} & \tilde{A}_{33} \end{bmatrix}$$

be the block partitioning of A_{22} consistent with that of the vectors u_2 and v_2. Using relations (2.13), we derive

$$\begin{aligned}
&\tilde{A}_{12}\tilde{u}_2 + \tilde{A}_{13}\tilde{u}_3 = 0, \\
&\tilde{A}_{22}\tilde{u}_2 + \tilde{A}_{23}\tilde{u}_3 = \rho(A)\tilde{u}_2, \\
&\tilde{A}_{12}^{\mathrm{T}}\tilde{v}_1 + \tilde{A}_{22}^{\mathrm{T}}\tilde{v}_2 = \rho(A)\tilde{v}_2, \\
&\tilde{A}_{13}^{\mathrm{T}}\tilde{v}_1 + \tilde{A}_{23}^{\mathrm{T}}\tilde{v}_2 = 0.
\end{aligned}$$

Since $\tilde{u}_2$ and $\tilde{v}_2$ are nontrivial and positive, the first and last of the latter equalities necessarily imply that

$$\tilde{A}_{12} = 0 \quad \text{and} \quad \tilde{A}_{23} = 0.$$

Therefore, from the remaining two equalities we have

$$\tilde{A}_{22}\tilde{u}_2 = \rho(A)\tilde{u}_2 \quad \text{and} \quad \tilde{A}_{22}^{\mathrm{T}}\tilde{v}_2 = \rho(A)\tilde{v}_2,$$

which completes the proof. □

Lemma 3. *Let A and B be $n \times n$ nonnegative matrices and let*

$$Au = \rho(A)u, \quad A^{\mathrm{T}}v = \rho(A)v, \tag{2.14}$$

$$Bz = \rho(B)z, \quad B^{\mathrm{T}}w = \rho(B)w. \tag{2.15}$$

Further, denote $C = A + B$, let

$$Cx = \rho(C)x, \tag{2.16}$$

where the vector x is nonnegative, and define

$$J = \{i : x_i \neq 0,\ 1 \leqslant i \leqslant n\}. \tag{2.17}$$

If all of the vectors u, v, z, and w are positive, then

$$\begin{aligned}
&C[J]x[J] = \rho(C)x[J], \\
&A[J]u[J] = \rho(A)u[J], \quad A[J]^{\mathrm{T}}v[J] = \rho(A)v[J], \\
&B[J]z[J] = \rho(B)z[J], \quad B[J]^{\mathrm{T}}w[J] = \rho(B)w[J].
\end{aligned} \tag{2.18}$$

Proof. In the case where the vector x is positive, we have nothing to prove. In the case where x contains zero components, we may assume (without loss of generality) that $J = \{k+1, \ldots, n\}, k \geqslant 1$, i.e., that the right Perron vector x of the matrix $C = A + B$ is of the form

$$x = \begin{bmatrix} 0 \\ x_2 \end{bmatrix},$$

where the subvector x_2 is positive. Let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \quad C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

be the block partitionings of the matrices A, B, and C induced by the above partitioning of the vector x. Then from (2.16) it readily follows that

$$C_{12}x_2 = 0,$$

whence, since x_2 is positive, we have

$$A_{12} = B_{12} = C_{12} = 0. \tag{2.19}$$

By using equalities (2.19) and (2.14), we easily obtain the relations

$$A_{11}u_1 = \rho(A)u_1, \quad A_{22}^{\mathrm{T}}v_2 = \rho(A)v_2, \tag{2.20}$$

where

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

are the corresponding partitionings of the positive vectors u and v. Taking into account relations (2.14), (2.19), (2.20), we derive

$$\begin{aligned} \rho(A)v^{\mathrm{T}}u = v^{\mathrm{T}}Au &= v_1^{\mathrm{T}}A_{11}u_1 + v_2^{\mathrm{T}}A_{22}u_2 + v_2^{\mathrm{T}}A_{21}u_1 \\ &= \rho(A)v^{\mathrm{T}}u + v_2^{\mathrm{T}}A_{21}u_1, \end{aligned}$$

which implies that $v_2^{\mathrm{T}}A_{21}u_1 = 0$. Since u_1 and v_2 are both positive, we conclude that necessarily

$$A_{21} = 0. \tag{2.21}$$

Similarly, we have

$$B_{21} = 0. \tag{2.22}$$

Thus, by (2.19), (2.21), and (2.22), the three matrices A, B, and C are actually block diagonal, and therefore relations (2.18), which, for $J = \{k + 1, \ldots, n\}$, take the form

$$C_{22}x_2 = \rho(C)x_2,$$

$$A_{22}u_2 = \rho(A)u_2, \quad A_{22}^{\mathrm{T}}v_2 = \rho(A)v_2,$$

$$B_{22}z_2 = \rho(B)z_2, \quad B_{22}^{\mathrm{T}}w_2 = \rho(B)w_2,$$

trivially stem from relations (2.16), (2.14), and (2.15), respectively. □

Proof of Theorem 5. First consider the case where both u and v are positive vectors and the matrix F has a positive right Perron vector x. Let the relations (2.3) hold. Then, for the matrix $C = DBD^{-1}$, we have

$$C(Du) = \rho(B)(Du), \quad (v^{\mathrm{T}}D^{-1})C = \rho(B)(v^{\mathrm{T}}D^{-1}). \tag{2.23}$$

Thus, $p = Du$ and $q = D^{-1}v$ are Perron vectors of C. Obviously,

$$p \circ q = u \circ v. \tag{2.24}$$

By relations (2.1), (2.23), and (2.24), we have

$$(Au) \circ v = (A^{\mathrm{T}}v) \circ u = \rho(A)u \circ v, \tag{2.25}$$

$$(Cp) \circ q = (C^{\mathrm{T}}q) \circ p = \rho(B)p \circ q = \rho(B)u \circ v. \tag{2.26}$$

If y is the positive vector defined by the relation

$$x \circ y = p \circ q = u \circ v, \tag{2.27}$$

then, applying Lemma 1 twice, we see that

$$y^{\mathrm{T}}Ax \geqslant v^{\mathrm{T}}Au = \rho(A)v^{\mathrm{T}}u \tag{2.28}$$

and

$$y^{\mathrm{T}}C^{\mathrm{T}}x = x^{\mathrm{T}}Cy \geqslant q^{\mathrm{T}}Cp = \rho(B)q^{\mathrm{T}}p = \rho(B)v^{\mathrm{T}}u. \tag{2.29}$$

Using equalities (2.27)–(2.29), we derive (2.6) as follows:

$$\rho(F)v^{\mathrm{T}}u = \rho(F)y^{\mathrm{T}}x = y^{\mathrm{T}}Fx = y^{\mathrm{T}}Ax + y^{\mathrm{T}}C^{\mathrm{T}}x$$
$$\geqslant [\rho(A) + \rho(B)]v^{\mathrm{T}}u.$$

Now let relations (2.4) be valid. Define the matrix

$$G = D_u D_v^{-1} B D_v D_u^{-1}, \tag{2.30}$$

where, for a vector $z = (z_i)$, $D_z = \mathrm{diag}(z)$ is the diagonal matrix the ith diagonal entry of which is equal to z_i. In view of (2.4), we obviously have

$$Gu = \rho(B)u, \quad G^{\mathrm{T}}v = \rho(B)v, \tag{2.31}$$

i.e., the matrix G satisfies relations (2.3). Therefore, as already established, for any diagonal matrix Δ with positive diagonal entries, we have

$$\rho(A + \Delta^{-1}G^{\mathrm{T}}\Delta) \geqslant \rho(A) + \rho(G) = \rho(A) + \rho(B),$$

provided that the matrix $A + \Delta^{-1}G^{\mathrm{T}}\Delta$ has a positive right Perron vector. Now it remains to note that if $\Delta = D_u^{-1}D_v D$, then

$$\Delta^{-1}G^{\mathrm{T}}\Delta = D^{-1}B^{\mathrm{T}}D \quad \text{and} \quad A + \Delta^{-1}G^{\mathrm{T}}\Delta = F.$$

This completes the proof of inequality (2.6) in the case where the vectors u, v, and x are all positive.

Now we consider the general case. If the vectors u and v have zero components, then by Lemma 2, for $I = \{i : u_i v_i \neq 0, 1 \leqslant i \leqslant n\}$, we have

$$\rho(A) = \rho(A[I]), \quad \rho(B) = \rho(B[I]), \tag{2.32}$$

whereas $A[I], B[I]$ and the positive vectors $u[I], v[I]$ satisfy the hypotheses of the theorem. Now let $\bar{x}$ be a right Perron vector of the matrix $F[I]$. Applying Lemma 3 to the matrices $A[I]$ and $D[I]^{-1}B^{\mathrm{T}}[I]D[I]$ (which have positive right and left Perron vectors), we arrive at the conclusion that there exists an index subset $J \subseteq I$ such that $\bar{x}[J]$ is a positive right Perron vector of the matrix $F[J]$, whereas

$$\rho(A[I]) = \rho(A[J]), \quad \rho(B[I]) = \rho(B[J]), \tag{2.33}$$

and, furthermore, the matrices $A[J]$ and $B[J]$ satisfy the hypotheses of the theorem with the positive vectors $u[J]$ and $v[J]$. By the already established part of this theorem, we thus have

$$\rho(A[J]) + \rho(B[J]) \leqslant \rho(F[J]).$$

On the other hand, in view of equalities (2.32) and (2.33),

$$\rho(A[J]) + \rho(B[J]) = \rho(A) + \rho(B)$$

whereas

$$\rho(F[J]) \leqslant \rho(F)$$

by the monotonicity of the Perron root with respect to principal submatrices (see, e.g., [2], Corollary 1.6). Thus, inequality (2.6) is proved completely.

Now consider the equality case. Assume that

$$\rho(A + D^{-1}B^{\mathrm{T}}D) = \rho(A) + \rho(B).$$

Since, by assumption, one of the matrices A, B is irreducible, the matrix F is also irreducible, and therefore its Perron vector x is positive as well as the vectors u and v. From the first part of the above proof it is then clear that the latter equality may occur under the assumption (2.3) if and only if simultaneously

$$y^{\mathrm{T}}Ax = \rho(A)v^{\mathrm{T}}u$$

and

$$x^{\mathrm{T}}DBD^{-1}y = \rho(B)v^{\mathrm{T}}u.$$

By Lemma 1, we then have

$$x = \gamma u, \quad \gamma > 0, \tag{2.34}$$

and

$$x = \zeta D^{-1}v, \quad \zeta > 0. \tag{2.35}$$

Using equality (2.34), we derive

$$\rho(F)u = Fu = Au + D^{-1}B^{\mathrm{T}}Du = \rho(A)u + D^{-1}B^{\mathrm{T}}Du,$$

which implies that Du is a left Perron vector for B. If B is irreducible, then necessarily Du is collinear to v. Similarly, from equality (2.35) it follows that $D^{-1}v$ is a right Perron vector for A, and thus if A is irreducible, then $D^{-1}v$ is collinear to u.

Finally, in the case where relations (2.4) hold, equality in (2.6) occurs if and only if

$$\rho(A + \Delta^{-1}G^{\mathrm{T}}\Delta) = \rho(A) + \rho(G),$$

where the matrix G is defined by (2.30), and relations (2.31) hold true. Therefore, as already shown, whenever either A or G is irreducible, the latter equality holds if and only if the vectors $\Delta u = D_u^{-1}D_v Du = Dv$ and v are collinear, i.e., if and only if $D = \alpha I_n$, where α is a positive constant. □

Remark 1. Actually, in the base case where both u and v are positive, F has a positive right Perron vector, and B satisfies relations (2.3), the result of Theorem 5 readily follows from Bapat's theorem. Indeed, by applying the similarity transformation with the matrix $D^{1/2}$ and Theorem 1 with $\alpha = 1/2$, we obtain

$$\rho(A + D^{-1}B^{\mathrm{T}}D) = \rho(\tilde{A} + \tilde{B}^{\mathrm{T}}) \geqslant \rho(\tilde{A}) + \rho(\tilde{B}) = \rho(A) + \rho(B),$$

where $\tilde{A} = D^{1/2}AD^{-1/2}$ and $\tilde{B} = D^{1/2}BD^{-1/2}$. We have preferred to present here a new independent proof for this case in order to make the whole proof more self-contained and less arithmetical than in [1].

Remark 2. It is pertinent to note that, in addition to extending Theorem 1, Theorem 5 also provides a generalization of Theorem 4 in [1] in the special case where $k = 2$ and $D^l = (E^l)^{-1}$, $l = 1, 2$. Indeed, if A satisfies the assumptions of Theorem 5, then, for $B = A^{\mathrm{T}}$, we have

$$\rho(A + D^{-1}AD) \geqslant 2\rho(A),$$

and, applying an appropriate diagonal similarity transformation, we see that the inequality

$$\rho(D_1^{-1}AD_1 + D_2^{-1}AD_2) \geqslant 2\rho(A)$$

holds for arbitrary diagonal matrices D_1 and D_2 with positive diagonal entries.

Setting, for $c > 0$ and $0 \leqslant \alpha \leqslant c$, $B = (c - \alpha)A$ and $B = (c - \alpha)A^{\mathrm{T}}$ in Theorem 5, we arrive at the following extension of Theorem 2.

Theorem 6. *Let A be an $n \times n$ nonnegative matrix. Then, for any $c > 0$, any α, $0 \leqslant \alpha \leqslant c$, and any diagonal matrix D with positive diagonal entries,*

$$c\rho(A) \leqslant \rho(\alpha A + (c - \alpha)D^{-1}AD) \tag{2.36}$$

and

$$c\rho(A) \leqslant \rho(\alpha A + (c - \alpha)D^{-1}A^{\mathrm{T}}D). \tag{2.37}$$

Further, if $0 < \alpha < c$ *and* A *is irreducible, then equality holds in* (2.36) *if and only if* $D = \gamma I_n$ *and in* (2.37) *if and only if* $D^{-1}A^{\mathrm{T}}D$ *and* A *have a common right Perron vector.*

Proof. Since the case $A = 0$ is trivial, assume that A is a nonzero matrix. For an irreducible A, the assertions of Theorem 6 are direct consequences of Theorem 5. So let A be reducible. Without loss of generality, we may assume that A is of the form

$$A = \begin{bmatrix} A_{11} & & \bigstar \\ & \ddots & \\ 0 & & A_{kk} \end{bmatrix},$$

where A_{ii}, $i = 1, \dots, k$, $k \geqslant 2$, are all irreducible, and for some r, $1 \leqslant r \leqslant k$, $\rho(A_{ii}) = \rho(A)$. Now, since the matrices

$$\alpha A_{ii} + (c - \alpha)D_i^{-1}A_{ii}D_i \quad \text{and} \quad \alpha A_{ii} + (c - \alpha)D_i^{-1}A_{ii}^{\mathrm{T}}D_i,$$

where

$$D = \begin{bmatrix} D_1 & & 0 \\ & \ddots & \\ 0 & & D_k \end{bmatrix}$$

is the corresponding block partitioning of D, are principal submatrices of the matrices $\alpha A + (c - \alpha)D^{-1}AD$ and $\alpha A + (c - \alpha)D^{-1}A^{\mathrm{T}}D$, respectively, the assertions of Theorem 6 in the case considered follow from those in the irreducible case by the monotonicity of the Perron root with respect to principal submatrices. □

In its turn, Theorem 6 implies the following direct generalization of Levinger's theorem.

Theorem 7. *Let* A *be an* $n \times n$ *nonnegative matrix and let* D *be a diagonal* $n \times n$ *matrix with positive diagonal entries. Then, for any* $c > 0$, *the function*

$$f(c, \alpha, D, A) = \rho(\alpha A + (c - \alpha)D^{-1}A^{\mathrm{T}}D), \quad 0 \leqslant \alpha \leqslant c, \tag{2.38}$$

as a function of α *is nondecreasing on* $[0, c/2]$ *and nonincreasing on* $[c/2, c]$.

Furthermore, if A *is irreducible, then the function* f *is constant on* $[0, c]$ *if and only if* A *and* $D^{-1}A^{\mathrm{T}}D$ *have a common right Perron vector.*

Proof. Denote

$$F_\alpha = \alpha A + (c-\alpha)D^{-1}A^{\mathrm{T}}D.$$

Since, obviously,

$$\rho(F_\alpha) = \rho(D^{-1}F_\alpha^{\mathrm{T}}D) = \rho(F_{c-\alpha}),$$

the function f is symmetric about the point $c/2$ with respect to α. Therefore, it is sufficient to show that f is nondecreasing on $[0,c/2]$. Let $0 \leqslant \alpha < \beta \leqslant c/2$. Consider the matrix

$$N(\alpha,\gamma) = \gamma F_\alpha + (1-\gamma)D^{-1}F_\alpha^{\mathrm{T}}D, \quad 0 \leqslant \gamma \leqslant 1.$$

By Theorem 6,

$$\rho(F_\alpha) \leqslant \rho(N(\alpha,\gamma)), \quad 0 \leqslant \gamma \leqslant 1. \tag{2.39}$$

A trivial calculation shows that

$$N(\alpha,\gamma) = (c-\alpha-\gamma c+2\alpha\gamma)A + (\alpha+\gamma c-2\alpha\gamma)D^{-1}A^{\mathrm{T}}D,$$

which implies that

$$N\left(\alpha, \frac{c-\alpha-\beta}{c-2\alpha}\right) = F_\beta. \tag{2.40}$$

Thus, in view of (2.39) and (2.40), we have

$$\rho(F_\alpha) \leqslant \rho(F_\beta), \quad 0 \leqslant \alpha < \beta \leqslant c/2.$$

Finally, if A is irreducible, then, by the first part of this theorem, the function f is constant on $[0,c]$ if and only if

$$f(c,0,D,A) = c\rho(A) = \rho\left(\frac{c}{2}A + \frac{c}{2}D^{-1}A^{\mathrm{T}}D\right) = f\left(c,\frac{c}{2},D,A\right),$$

which, by Theorem 6, may occur if and only if A and $D^{-1}A^{\mathrm{T}}D$ have a common right Perron vector. □

Remark 3. It should be mentioned that in fact inequality (2.40) readily follows from Theorem 3 if one takes into account the trivial identities

$$f(c,\alpha,D,A) = f(c,\alpha,I,D^{1/2}AD^{-1/2}),$$

$$f(c,\alpha,I,A) = f(1,\alpha/c,I,cA).$$

As in the case of Theorem 5, we have again preferred to provided the self-contained proof of the generalized result, which is similar to that in [1] up to incorporating the matrix D. Another reason for presenting the derivation of Theorem 7 from Theorem 6 is to demonstrate the similarity of the proofs of Theorem 7 and Theorem 9 (see Section 3).

We conclude this section by presenting a useful and seemingly new reformulation of Levinger's theorem that exhibits the dependence of the Perron root on scaling the skew-symmetric part of a nonnegative matrix with a real number $\beta, |\beta| < 1$.

Theorem 8. *Let $S = S^{\mathrm{T}}$ and $C = -C^{\mathrm{T}}$ be two $n \times n$ matrices and assumed that the matrix $S + C$ is nonnegative. Then the function*

$$h(\beta) = \rho(S + \beta C), \quad -1 \leqslant \beta \leqslant 1,$$

is nondecreasing on $[-1, 0]$ and nonincreasing on $[0, 1]$. Furthermore, $h(\beta)$ is constant on $[-1, 1]$ if and only if $S + C$ and $S - C$ have a common right Perron vector.

Proof. Denote $A = S + C$. Then A is a nonnegative matrix and $A^{\mathrm{T}} = S - C$. For any $\alpha, 0 \leqslant \alpha \leqslant 1$, we have

$$\alpha A + (1 - \alpha)A^{\mathrm{T}} = S + (2\alpha - 1)C = S + \beta C, \quad \beta = 2\alpha - 1,$$

and the result stems from Levinger's theorem. □

3. The multiplicative counterpart of generalized Levinger's theorem

In this section, we prove the following multiplicative counterpart of Theorem 7.

Theorem 9. *Let A be an $n \times n$ nonnegative matrix, D be an $n \times n$ diagnal matrix with positive diagonal entries, and let $c \geqslant 1$. Then the function*

$$g(c, \alpha, D, A) = \rho(A^{(\alpha)} \circ (D^{-1}A^{\mathrm{T}}D)^{(c-\alpha)}), \quad 0 \leqslant \alpha \leqslant c, \tag{3.1}$$

as a function of α in nonincreasing on $[0, c/2]$ and nondecreasing on $[c/2, c]$.

Furthermore, if A is irreducible and $c = 1$, then the function $g(1, \alpha, D, A)$ is constant on $[0, 1]$ if and only if $A = \Delta^{-1}A^{\mathrm{T}}\Delta$, where Δ is a diagonal matrix with positive diagonal entries.

Proof. Taking $B = D^{-1}A^{\mathrm{T}}D$ in Theorem 4, we arrive at the inequality

$$\rho(A^{(\alpha)} \circ (D^{-1}A^{\mathrm{T}}D)^{(c-\alpha)}) \leqslant \rho(A)^{c}, \quad 0 \leqslant \alpha \leqslant c. \tag{3.2}$$

Further, in accordance with Theorem 4 of this paper and Lemma 2 in [6], equality in (3.2) occurs if and only if there exists a nonempty subset $I \subseteq \{1, \ldots, n\}$ such that

(i) $A[I]$ is irreducible,

(ii) $\rho(A[I]) = \rho(A)$,

(iii) $A[I] = \Delta^{-1}A^{\mathrm{T}}[I]\Delta$, where Δ is a diagonal matrix with positive diagonal entries.

Now we consider the function (3.1). Since, obviously,

$$g(c,\alpha,D,A) = g(c,c-\alpha,D,D^{-1}A^{\mathrm{T}}D),$$

it is sufficient to establish that g is a nonincreasing function of α on $[0,c/2]$. So let $0 \leqslant \alpha < \beta \leqslant c/2$. Under the notation

$$F_\alpha = A^{(\alpha)} \circ (D^{-1}A^{\mathrm{T}}D)^{(c-\alpha)},$$

we must ascertain that

$$\rho(F_\alpha) \geqslant \rho(F_\beta). \tag{3.3}$$

In view of (3.2), for the matrix

$$N(\alpha,\gamma) = F_\alpha^{(\gamma)} \circ (D^{-c}F_\alpha^{\mathrm{T}}D^{c})^{(1-\gamma)}, \quad 0 \leqslant \gamma \leqslant 1,$$

we have

$$\rho(N(\alpha,\gamma)) \leqslant \rho(F_\alpha), \quad 0 \leqslant \gamma \leqslant 1.$$

On the other hand, it can be straightforwardly verified that

$$N\left(\alpha, \frac{c-\alpha-\beta}{c-2\alpha}\right) = F_\beta.$$

This completes the proof of inequality (3.3).

Finally, the function $g(1,\alpha,D,A)$ is constant on $[0,1]$ if and only if

$$\rho(A) = g(1,0,D,A) = g(1,1/2,D,A) = \rho(A^{(1/2)} \circ (D^{-1}A^{\mathrm{T}}D)^{(1/2)}).$$

But, for an irreducible matrix A, this can occur if and only if A and A^{T} are diagonally similar. $\square$

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